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AND BIASED RELAY SYSTEM

by

Arthur L. Schoenstadt

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NAVAL POSTGRADUATE SCHOOL
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ABSTRACT:

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where $\beta>0$, and $\alpha_2 \geq 0$. (This second condition is necessary to avoid "chatter" solutions.) Such a system is referred to as biased when $c \neq 0$, and unbiased when $c = 0$. In a preliminary part, a geometrical construction for the necessary and sufficient condition is derived. The construction is then used to show that existence of a limit cycle in the unbiased case is a necessary and almost sufficient condition for the existence of a cycle in the biased case, and, further, that a bifurcation into exactly two limit cycles will occur for certain values of c.

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solutions.) Such a system is referred to as biased when $c \neq 0$, and unbiased when $c = 0$. In a preliminary part, a geometrical construction for the necessary and sufficient condition is derived. The construction is then used to show that existence of a limit cycle in the unbiased case is a necessary and almost sufficient condition for the existence of a cycle in the biased case, and, further, that a bifurcation into exactly two limit cycles will occur for certain values of c.

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I. INTRODUCTION

Consider the two dimensional vector ordinary differential equation:

$$\dot{\underline{v}} = \underline{\underline{A}}\, \underline{v} + \underline{u}_0 \operatorname{sgn}(rv_1 + sv_2 - c) , \tag{1}$$

where $r^2 + s^2 > 0$, $\underline{u}_0$ is some non-zero constant vector, and $\underline{\underline{A}}$ is a constant matrix whose characteristic polynomial is

$$\lambda^2 + 2\gamma\lambda + (\gamma^2 + \omega^2) , \quad \gamma > 0 .$$

Equations of this type commonly arise in systems involving ideal relays. We refer to the curve $rv_1 + sv_2 - c = 0$ as the switching line in these applications, and to the cases $c \neq 0$ and $c = 0$ as biased and unbiased, respectively. In this paper, we investigate necessary and sufficient conditions under which (1) will have periodic (limit cycle) solutions. We shall show that existence of a limit cycle in the unbiased case is a necessary and almost sufficient condition for existence of a limit cycle in the biased case, however, that not all biasings (i.e., values of $c \neq 0$) will produce cycles. We also develop an extremely powerful and interesting geometrical interpretation of the necessary and sufficient conditions, and use this interpretation to demonstrate a bifurcation into two cycles for the same value of c under certain circumstances.

Equations of this type have been considered by Fleishman [2], Davis and Fleishman [1], and Flügge-Lotz [3], among others. However, Davis and Fleishman consider primarily the special cases when only even order derivatives are present, and Flügge-Lotz does not give a complete

characterization of the limit cycles. Also, neither author considers the question of multiple limit cycles in the general case.

For the purposes of our investigation, it is convenient to introduce a non-singular linear transformation and time scaling (Appendix A) that reduces (1) to:

$$\dot{x}_1 = -2\beta x_1 - (\beta^2 + 1)x_2 + \alpha_1 \operatorname{sgn}(x_2 - c) ,$$

$$\dot{x}_2 = x_1 + \alpha_2 \operatorname{sgn}(x_2 - c) , \qquad (2)$$

where $\beta > 0$ and $\alpha_1^2 + \alpha_2^2 > 0$.

This now represents a damped system, forced by a relay switching along the transformed switching curve $x_2 = c$. It can be shown that a necessary and sufficient condition to avoid the phenomenon called "chatter" in (2), and to insure that a solution will exist at each point in the (x_1, x_2) plane, is $\alpha_2 \geq 0$. We now make that restriction. Furthermore, (2) involves only odd functions of x_1 and x_2 ; hence we can assume, without loss of generality, that $c \geq 0$.

II. THE PHASE PLANE

Our analysis of (2) begins with the observation that, viewed separately in the regions $x_2 > c$ and $x_2 < c$, (2) reduces to a linear, constant coefficient system. Elementary methods then yield the solutions:

$$\begin{aligned} x_1^*(t) &= -c_1^* e^{-\beta t}(\beta \cos t + \sin t) + c_2^* e^{-\beta t}(\cos t - \beta \sin t) - \alpha_2 , \\ x_2^*(t) &= c_1^* e^{-\beta t}\cos t + c_2^* e^{-\beta t}\sin t + \frac{(2\beta\alpha_2 + \alpha_1)}{(\beta^2 + 1)} \end{aligned} \tag{3a}$$

in the region $x_2(t) > c$, and

$$\begin{aligned} x_1(t) &= -c_1 e^{-\beta t}(\beta \cos t + \sin t) + c_2 e^{-\beta t}(\cos t - \beta \sin t) + \alpha_2 , \\ x_2(t) &= c_1 e^{-\beta t}\cos t + c_2 e^{-\beta t}\sin t - \frac{(2\beta\alpha_2 + \alpha_1)}{(\beta^2 + 1)} \end{aligned} \tag{3b}$$

for $x_2(t) < c$. Thus the trajectories in the phase plane are simply (arcs of) decaying (since $\beta > 0$) counterclockwise spirals, centered at

$$\left(-\alpha_2 , \frac{(2\beta\alpha_2 + \alpha_1)}{(\beta^2 + 1)}\right) \tag{4}$$

for the region $x_2 > c$, and at

$$\left(\alpha_2 , -\frac{(2\beta\alpha_2 + \alpha_1)}{(\beta^2 + 1)}\right) \tag{5}$$

for the region $x_2 < c$. These centers will be asymtotically stable critical points if and only if they are actually above and below $x_2 = c$, respectively, e.g., (4) describes a critical point if and only if

$$\frac{2\beta\alpha_2 + \alpha_1}{(\beta^2 + 1)} > c \ .$$

Note that, if and only if $c = 0$, the origin is also a critical point, but obviously an unstable one.

A representative view of the phase plane behavior can be obtained by considering the region $x_2 > c$. There, in the case

$$(2\beta\alpha_2 + \alpha_1) \leq c(\beta^2 + 1) \ ,$$

the center for trajectory arcs is located in the region $x_2 \leq c$, with an x_1 co-ordinate of $-\alpha_2 \leq 0$. Thus, because the spiral arcs are counterclockwise, every arc emanating from a point in $x_2 > c$ intersects the switching line at an $x_1 < 0$ in some finite time, and exits the region. Furthermore, since the angular frequency is unity, then each arc spends no more time than π in the region, with a time equal to π occurring only when the center of the arcs is located on the switching line (i.e., when $(2\beta\alpha_2 + \alpha_1) = c(\beta^2 + 1)$).

The situation when

$$(2\beta\alpha_2 + \alpha_1) > c(\beta^2 + 1)$$

is significantly different in that the spirals are now centered at a stable critical point actually located in the region $x_2 > c$. Thus, some trajectories never exit the region, but decay down to the critical point, while others intersect the switching line and exit the region.

The "boundary" between the arcs that exit the region and those that decay to the critical point is the arc that terminates on the switching line, directly below the critical point, i.e., at

$$(-\alpha_2, c) \quad . \tag{6}$$

This arc, which also exits the region, is denoted (*) in Figure 1. We let T_o^* denote the time represented by this arc, where clearly $\pi < T_o^* < 2\pi$. It is fairly easily shown that every other arc which exits the region $x_2 > c$ represents less time in the region than T_o^* , i.e., T_o^* is the maximum time an arc can spend in $x_2 > c$ and still exit there.

T_o^* can be determined, using $(x_1^*(t), x_2^*(t))$ to denote the solution in the region $x_2 > c$, by noting that the condition described by equation (6) and curve (*) of Figure 1 is equivalent to

$$\begin{aligned} x_1^*(T_o^*) &= -\alpha_2 \\ x_2^*(T_o^*) &= x_2^*(0) = c \quad . \end{aligned} \tag{7}$$

Using (3) and simplifying the resulting three equations, we arrive at:

$$e^{-\beta T_o^*} = \cos T_o^* - \beta \sin T_o^* \ , \tag{8}$$

which can easily be shown to have one and only one solution in $(\pi, 2\pi)$, hence T_o^* is uniquely defined.

The analysis of the region $x_2 < c$ is similar, and so is omitted. The only point of interest in this case is that, when, a critical point is located in this region (i.e., $-(2\beta\alpha_2 + \alpha_1) < c(\beta^2 + 1)$) a "maximum time", T_o can be defined, similar to (7) by:

$$\begin{aligned} x_1(T_o) &= \alpha_2 \\ x_2(T_o) &= x_2(0) = c \quad . \end{aligned} \tag{9}$$

Straighforward calculation shows T_o satisfies (8), hence, since the solution is unique,

$$T_o = T_o^* \quad . \tag{10}$$

(For the remainder of this discussion, we drop further reference to T_o^*, replacing it by T_o .) Again we can show that no curve which exits $x_2 < c$ can spend more than T_o time there.

III. THE CONDITIONS FOR A LIMIT CYCLE

To derive the necessary and sufficient conditions for the existence of a limit cycle solution to (2), we begin with the solutions as given by (3). We let T_1 denote the time spent by one trajectory arc in $x_2 < c$, and T_2 the time on an arc in $x_2 > c$. Then, since the system (2) is autonomous, and since trajectories must be continuous in the (x_1, x_2) plane, it follows that a necessary and sufficient condition for a limit cycle of period $(T_1 + T_2)$ to exist is:

$$\left.\begin{aligned} x_1(T_1) &= x_1^*(0) \\ x_1(0) &= x_1^*(T_2) \\ x_2(0) &= x_2(T_1) = c \\ x_2^*(0) &= x_2^*(T_2) = c \end{aligned}\right\} \quad T_1, T_2 \leq T_o . \tag{11}$$

where it is understood $x_1(0)$ represents $x_1(0^+)$, etc.

The condition given by (11) can be reduced by using the solutions given by (3) and simplifying the resulting expressions by eliminating c_2 and c_2^*. This leads, in the case where neither T_1 nor T_2 equal π, to the equations:

$$\left.\begin{aligned} c_1 \frac{\cosh \beta T_1 - \cos T_1}{\sin T_1} + c_1^* \frac{\cosh \beta T_2 - \cos T_2}{\sin T_2} &= 0 , \quad (12) \\ c_1 \frac{\sinh \beta T_1}{\sin T_1} - c_1^* \frac{\sinh \beta T_2}{\sin T_2} &= -2 \frac{(1-\beta^2)\alpha_2 - \beta\alpha_1}{(\beta^2 + 1)} , \quad (13) \end{aligned}\right\} \begin{aligned} T_1 &\leq T_o \\ T_2 &\leq T_o \end{aligned}$$

where

$$c_1 - c_1^* = 2 \frac{(2\beta\alpha_2 + \alpha_1)}{(\beta^2 + 1)} , \tag{14}$$

and the value of c for which this cycle occurs is given by

$$c = \frac{1}{2}(c_1 + c_1^*) \quad . \tag{15}$$

These equations express the necessary and sufficient conditions for existence of a limit cycle, except in the cases noted. (Note these later cases correspond respectively to $c_1 = 0$, if and only if, $T_1 = \pi$ and $c_1^* = 0$, if and only if, $T_2 = \pi$.) In these cases, expressions similar to (12)-(14) can be derived. For example, in the case $c_1 = 0$, $c_1^* \neq 0$, the condition for existence of a limit cycle of period $(T_2 + \pi)$ is:

$$- \frac{\sinh \beta\pi}{\cosh \beta\pi + 1} \left[\frac{\cosh \beta T_2 - \cos T_2}{\sin T_2} \right]$$

$$= \frac{\sinh \beta T_2}{\sin T_2} + \frac{(1-\beta^2)\alpha_2 - 2\beta\alpha_1}{(2\beta\alpha_2 + \alpha_1)} \quad , \; T_2 \leq T_o \quad . \tag{16}$$

We now develop an interesting, and powerful, geometric interpretation of the necessary and sufficient conditions when $c_1^2 + c_1^{*2} > 0$. Consider (12-(14), which represent the necessary and sufficient conditions when $c_1 c_1^* \neq 0$, and let

$$F(t) = \frac{\sinh \beta t}{\sin t} \quad , \quad G(t) = \frac{\cosh \beta t - \cos t}{\sin t} \tag{17}$$

Then (12) - (14) is equivalent to the vector equation

$$c_1 \begin{bmatrix} G(T_1) \\ F(T_1) \\ 1 \end{bmatrix} - c_1^* \begin{bmatrix} -G(T_2) \\ F(T_2) \\ 1 \end{bmatrix} - \frac{2(2\beta\alpha_2 + \alpha_1)}{(\beta^2 + 1)} \begin{bmatrix} 0 \\ -\dfrac{(1-\beta^2)\alpha_2 - \beta\alpha_1}{(2\beta\alpha_2 + \alpha_1)} \\ 1 \end{bmatrix} = 0 , \tag{18}$$

if $(2\beta\alpha_2 + \alpha_1) \neq 0$. But this means that the three vectors in (18) are linearly dependent, i.e., lie in the same plane in R^3 . Clearly this plane cannot coincide with the $z = 1$ plane, yet since each vector in (18) terminates on the $z = 1$ plane, linear dependence then requires that the end points of the three vectors lie in a line in that plane. Thus, (12)-(14) can be satisfied if and only if the vectors

$$\begin{bmatrix} G(T_1) \\ F(T_1) \end{bmatrix} , \quad \begin{bmatrix} -G(T_2) \\ F(T_2) \end{bmatrix} , \quad \text{and} \quad \begin{bmatrix} 0 \\ \dfrac{(1-\beta^2)\,\alpha_2 - \beta\alpha_1}{-(2\beta\alpha_2 + \alpha_1)} \end{bmatrix} \tag{19}$$

lie in the same straight line. But by drawing the parametric curves:

$$\zeta_1 = \{(G(\tau),\ F(\tau))\ 0 \leq \tau \leq T_o\} , \tag{20}$$
$$\zeta_2 = \{(-G(\tau), F(\tau))\ 0 \leq \tau \leq T_o\}$$

plus the point

$$P_o = (0, -\frac{(1-\beta^2)\,\alpha_2 - \beta\alpha_1}{(2\beta\alpha_2 + \alpha_1)}) , \quad (2\beta\alpha_2 + \alpha_1) \neq 0 , \tag{21}$$

we have that (provided $c_1 c_1^* \neq 0$, and $(2\beta\alpha_2 + \alpha_1) \neq 0)$, any line

through P_o and both curves generates a limit cycle, and, conversely, for any limit cycle, there must be a straight line through the point and both curves.

Straighforward but laborious calculation, plus liberal application of hyperbolic identities, shows that both $F(\tau)$ and $G(\tau)$ are positive and monotonically increasing on $(0,\pi)$, and negative and convex down on $(\pi, 2\pi)$. Furthermore $G(\tau)$ has its maximum to the right of T_o (i.e. $G(\tau)$ increases monotonically on (π, T_o)) , while $F(\tau)$ has its maximum at a $T_m < T_o$. Also, ζ_1 and ζ_2 are convex up in the upper half plane, intersecting at $(0,\beta)$, and convex down in the lower half plane, being asymptotic to the lines

$$u = \frac{\sinh \beta\pi}{\cosh \beta\pi + 1} w - \beta \quad , \tag{22}$$

and

$$u = - \frac{\sinh \beta\pi}{\cosh \beta\pi + 1} w - \beta \tag{23}$$

respectively, as $\tau \to \pi$ $(w \to \pm \infty)$. Thus ζ_1 and ζ_2 must have the approximate shapes shown in Figure 2. Figure 3 shows one situation which would lead to a limit cycle. Observe from (12) that the ratio of the horizontal intersections on ζ_1 and ζ_2 , which we denote:

$$\rho = - \frac{G(T_1)}{G(T_2)} = \frac{c_1^*}{c_1} \quad , \tag{24}$$

satisfies (from (14) and (15));

$$c = \frac{1}{2} (c_1 + c_1^*) = \frac{(2\beta\alpha_2 + \alpha_1)}{(\beta^2 + 1)} \frac{(1 + \rho)}{(1 - \rho)} \quad . \tag{25}$$

Thus, since ρ is just the ratio of horizontal co-ordinates, we could easily compute, from the geometrical configuration, the value of c for the cycle shown in Figure 3. (Assuming values for β_1, α_1 and α_2 are known.)

Equations (24)-(25) will be especially useful in discussing the range of values of c for which cycles can occur, and, since it will become important later, note that on any interval not including $\rho = 1$, c as given by (25) is a monotonically increasing [decreasing] function of ρ for $(2\beta\alpha_2 + \alpha_1) > 0$ $[(2\beta\alpha_2 + \alpha_1) < 0]$.

In the discussion so far, this geometrical interpretation has been limited to the cases $c_1 c_1^* \neq 0$, and $(2\beta\alpha_2 + \alpha_1) \neq 0$. However, the interpretation can easily be extended to all cases where $c_1^2 + c_1^{*2} > 0$ by extending the interpretation of a line intersecting the point and both curves to allow intersection in the limit at infinity. Thus, for example, the case $c_1 = 0$, $c_1^* \neq 0$ and $(2\beta\alpha_2 + \alpha_1) \neq 0$ is treated (refer to (24)) as the limiting case

$$G(T_1) \to \pm \infty \quad .$$

Now, referring to Figures 2-3, it is obvious "intersection" in this limiting case occurs if and only if the straight line is through P_o , parallel to the asymptote of ζ_1 , intersecting ζ_2 . (Figure 4). That is, there must be a T_2 such that

$$\frac{F(T_2) + \dfrac{(1 - \beta^2)\,\alpha_2 - \beta\alpha_1}{(2\beta\alpha_1 + \alpha_2)}}{- G(T_2)} = \frac{\sinh \beta\pi}{\cosh \beta\pi + 1} \quad . \qquad (26)$$

But referring to the definitions of $F(t)$ and $G(t)$ it can be seen (26)

is identical to (16). A similar process can be followed when $c_1^* = 0$, $c_1 \neq 0$, and $(2\beta\alpha_2 + \alpha_1) \neq 0$.

Lastly, interpreting (19)-(21) where $(2\beta\alpha_2 + \alpha_1) = 0$ as the limiting case when P_o approaches the point at infinity along the vertical axis, we see that an "intersection" can occur if and only if the points on ζ_1 and ζ_2 lie on the same vertical line, or equivalently $G(T_1) = - G(T_2)$. However, since by (14)-(15) this case corresponds to $c_1 = c_1^* = c$, this interpretation immediately satisfies (12), and yields from (13) a cycle occuring for every value of c satisfying:

$$c = - 2\alpha_2 \left\{ \frac{\sinh \beta T_1}{\sin T_1} - \frac{\sinh \beta T_2}{\sin T_2} \right\}^{-1} , \quad (27)$$

where T_1 and T_2 are solutions of $G(T_1) = -G(T_2)$, and the absolute value of the quantity in brackets represents the vertical distance between the two points of intersection.

Thus, including the limiting cases, we see that for all $c_1^2 + c_1^{*2} > 0$, the geometrical interpretation is equivalent to the solution of (12)-(14), and hence it expresses the necessary and sufficient condition for existence of a limit cycle.

IV. EXISTENCE OF LIMIT CYCLES

In this section we apply the geometric construction developed above to derive necessary and sufficient conditions under which a limit cycle(s) will exist. Our main conclusion will be to determine existence conditions in terms of β , α_1 and α_2 . Specifically, we show a bound, dependent solely on β , such that no cycles exist for any values of c if the ratio $[\alpha_1/\alpha_2]$ exceeds this bound, and, for all values less than this bound, there is a $c_{max} > 0$, depending on β , α_1 , and α_2 such that a cycle will exist if and only if $|c| \leq c_{max}$.

We begin by noting the following result, alluded to in the introduction to this paper:

Lemma 1: A limit cycle will exist for values of $c > 0$ if and only if a cycle of the same period exists for $-c$.

The proof of this, which follows straightforwardly from the oddness of all functions in (2), is omitted here.

The following theorem gives the condition under which no cycles of any period are possible, irrespective of the biasing.

<u>Theorem 1:</u> There is an $\eta_{max}(\beta)$ such that (2) will have no limit cycle solutions if either

(a) $\alpha_2 = 0$, or

(b) $\alpha_2 \neq 0$ and $\left[\dfrac{\alpha_1}{\alpha_2}\right] > \eta_{max}(\beta)$.

$$\underline{\text{Proof:}}\quad \text{Let}\quad M(\beta) = -\max_{(\pi,2\pi)} \left[\frac{\sinh \beta t}{\sin t}\right] > 0, \tag{28}$$

and

$$\eta_{max}(\beta) = \frac{1 - \beta^2 - 2\beta M}{(\beta + M)} > -2\beta \quad . \tag{29}$$

(Note - M is the maximum vertical coordinate on both ζ_1 and ζ_2 in the lower half plane.) It is then easily shown that conditions (a) and (b) are equivalent to

$$- M < - \frac{(1 - \beta^2)\alpha_2 - \beta\alpha_1}{(2\beta\alpha_2 + \alpha_1)} \leq \beta \quad .$$

However, this implies that P_o (as given by (21)) lies above the line joining the maxima of ζ_1 and ζ_2 in the lower half plane, but at or below the juncture of ζ_1 and ζ_2 in the upper half plane (Figure 5). The convexity of ζ_1 and ζ_2 and the quadrants in which the curves are located immediately imply no straight line through P_o can intersect both ζ_1 and ζ_2, hence no cycle can exist. ∎

Note that, by defining

$$\eta = \left[\frac{\alpha_1}{\alpha_2}\right] \quad , \quad \alpha_2 \neq 0 \ , \tag{30}$$

P_o can be equivalently defined

$$P_o(\beta,\eta) = \left(0, - \frac{(1 - \beta^2) - \beta\eta}{(2\beta + \eta)}\right) \quad . \tag{31}$$

Then, if the situation when $\alpha_2 = 0$ is considered as the limiting case $\eta \to \pm \infty$, the previous theorem implies cycles can exist only if $- \infty < \eta \leq \eta_{max}(\beta)$. We now complete the result by showing that if $- \infty < \eta \leq \eta_{max}(\beta)$, a limit cycle will exist for at least one value of c.

Theorem 2: If $\eta = \eta_{max}(\beta)$, a cycle will exist if and only if $c = 0$.

Proof: If $\eta = \eta_{max}(\beta)$, then

$$P_o = (0,-M) ,$$

that is P_o lies on the line joining the maxima ζ_1 and ζ_2 in the lower half plane, (Figure 5). But the convexity of ζ_1 and ζ_2 imply this is also the only line through P_o intersecting both curves. Thus one and only one cycle will occur in this case. From Lemma 1, $c = 0$. $\blacksquare$

The above result is the reason why, in the introduction, we commented that existence of a limit cycle in the unbiased case was a necessary and almost sufficient to give cycles for some biasings, for, as the following shows, only when $\eta_{max}(\beta) = \eta$ can a cycle exist only for $c = 0$.

Theorem 3: If $-\infty < \eta < \eta_{max}(\beta)$, then there is a $c_{max} > 0$, such that a limit cycle will exist if and only if

$$|c| \leq c_{max} .$$

Proof: By Lemma 1 we restrict ourselves to $c \geq 0$. The actual proof is divided into three subcases; $-2\beta < \eta < \eta_{max}(\beta)$, $\eta = -2\beta$, and $-\infty < \eta < -2\beta$.

Case I: The case $-2\beta < \eta < \eta_{max}(\beta)$, corresponds (with our restriction $\alpha_2 \geq 0$) to $(2\beta\alpha_2 + \alpha_1) > 0$, and so, by (25), $c > 0$ corresponds to $-1 \leq \rho < 1$. Also, from (29) - (31), we can show that in this case $P_o(\eta,\beta)$ is located in the lower half plane, below the line joining the maxima there of ζ_1 and ζ_2 . However,

with P_o so located, we can define θ_o, $0 < \theta_o < \frac{\pi}{2}$, as the largest clockwise angle from the horizontal for which a straight line through P_o intersects ζ_1 in the third quadrant. Then clearly any line through P_o with slope, m_σ, will intersect both ζ_1 and ζ_2 if and only if

$$|m| \leq \tan \theta_o \quad . \qquad \text{(Figure 6)}$$

However, since ρ must vary continuously with m_σ (along lines intersecting both curves), and since ρ can approach 1 only if the intersecting line approaches the vertical, then ρ must be bounded away from 1, and there exists a ρ_{max} such that intersection will occur for $c > 0$ if and only if $-1 \leq \rho \leq \rho_{max} < 1$. It follows from (25) that a cycle will exist for $c \geq 0$ if and only if

$$0 \leq c \leq c_{max} = \frac{(2\beta\alpha_2 + \alpha_1)}{(\beta^2 + 1)} \quad \frac{(1 + \rho_{max})}{(1 - \rho_{max})} \quad ,$$

since the right hand side of (25) is strictly increasing for $(2\beta\alpha_2 + \alpha_1) > 0$ and $-1 \leq \rho < 1$. Clearly c_{max} depends on β, α_1 and α_2.

Case II: When $-2\beta = \eta$, $(2\beta\alpha_2 + \alpha_1) = 0$, and as we have discussed before in this case limit cycles exist if and only if c satisfies (27), where in (27), the quantity:

$$\left\{ \frac{\sinh \beta T_1}{\sin T_1} - \frac{\sinh \beta T_2}{\sin T_2} \right\}$$

represents the distance between ζ_1 and ζ_2 on any vertical line

intersecting both (since $P_o(\eta)$ is imagined passed to a limit at infinity along the vertical axis). Since the distance must clearly have a non-zero minimum, and since (27) involves its reciprocal, it follows straightforwardly that there is a c_{max} such that c, as defined by (27) satisfies

$$0 \le c \le c_{max} \quad .$$

<u>Case III</u>: When $-\infty < \eta < -2\beta$, the proof is essentially similar to case I, except $P_o(\eta,\beta)$ is located in the upper half-plane, above the juncture of ζ_1 and ζ_2. $\blacksquare$

Theorem 3 completes the description of the necessary and sufficient conditions under which a limit cycle(s) will exist. Note that although this theorem predicts at least one cycle for every value of c satisfying $0 \le |c| \le c_{max}$, it does not address the possibility of more than one cycle occuring at a single value of c. We shall pursue this in the next section. However, we first investigate in some more depth the properties of $\eta_{max}(\beta)$.

Elementary methods show that, for a given β, $M(\beta)$ as defined by (29) is given by

$$M(\beta) = -\frac{\sinh \beta T_m(\beta)}{\sin T_m(\beta)}$$

where $T_m(\beta)$ is the solution, in $(\pi, 2\pi)$, of the equation

$$\tan t = \frac{1}{\beta} \tanh \beta t \quad .$$

Since the right hand side of this equation can be shown to be monotonically decreasing in β (for fixed t), it follows that $T_m(\beta)$ is monotonically decreasing in β, and

$$\lim_{\beta\to\infty} T_m(\beta) = \pi \quad .$$

Further direct calculation will show

$$\frac{dM(\beta)}{d\beta} = - \frac{T_m(\beta) \cosh \beta T_m(\beta)}{\sin T_m(\beta)} > 0$$

since $\pi < T_m$. Thus $M(\beta)$ is a monotonically increasing, and clearly $M(\beta) \to \infty$ as $\beta \to \infty$.

With $M(\beta)$ thus fully characterized, it is easily shown that $\eta_{max}(\beta)$, as given by (29), is a monotonically decreasing function of β , asymptotic to the curve $\eta = -2\beta$ as $\beta \to \infty$. Figure 7 shows the general dependence of η_{max} , and it is fairly obvious that

$$\eta_{max}(\beta) \doteq -2\beta \;, \quad \beta > 1 \quad .$$

V. BIFURCATION OF THE LIMIT CYCLES

The previous section showed that for $-\infty < \eta \leq \max(\beta)$, there was a $c_{max}(\eta,\beta)$ such that at least one limit cycle would exist for every c satisfying $0 \leq |c| \leq c_{max}$. In this section we complete the description of the limit cycles by showing there are values of c that allow more than one cycle, how these values are determined, and how the transition from single to multiple cycles occurs.

Consider the situation represented by figure 8. When the straight line intersects two curves ζ_1 and ζ_2 , the horizontal co-ordinates of the intersections and slopes there of these curves are denoted w_1 , m_1 and w_2 , m_2 respectively, and σ is the angle between the line and the horizontal. Then, using elementary calculus, it can be shown that:

$$\frac{d}{d\sigma}\left\{\frac{w_1}{w_2}\right\} = \frac{w_1}{w_2}\left\{m_\sigma^{\,2} + 1\right\} \frac{(m_2-m_1)}{(m_1-m_\sigma)\,(m_2-m_\sigma)}$$

However, referring to (24), this equation becomes

$$\frac{d\rho}{d\sigma} = \rho\,(m_\sigma^{\,2} + 1)\,\frac{(m_2-m_1)}{(m_1-m_\sigma)\,(m_2-m_\sigma)} \tag{33}$$

where m_σ is the slope of the intersecting line. This equation allows us to determine, relatively easily, the points where $\frac{d\rho}{d\sigma}$ changes sign. But from (25), c is a monotonically increasing function of ρ for $(2\beta + \eta) < 0$, and a monotonically decreasing function for $(2\beta + \eta) < 0$. Thus every time $\frac{d\rho}{d\sigma}$ changes sign (for fixed values of β ,η) , a bifurcation occurs, i.e. two cycles exist for the same values of c .

With this formulation, and our main intent being to characterize the algebraic sign of $\frac{d\rho}{d\sigma}$ as given by (33), we proceed. As will emerge

more clearly later in the discussion, there are several special subcases, of increasing complexity, that must be considered. We shall not do this exhaustively, but essentially look in full detail only at one of these cases, then comment on the others. In essence, the complex cases arise when P_o is located "near" $(0,-M)$, so that a single line through P_o can intersect ζ_1 (or ζ_2) twice in the same quadrant.

Consider the case where P_o is located in the lower half plane, and at or below the intersection of the tangent to ζ_1 at T_o with the verticle axis. (Figure 9). Since we can show $m_1(T_o) = -1$, this condition can be reduced, using the definition of T_o , to

$$0 < 2\beta + \eta \leq - \frac{(1 + \beta^2)}{2(F(T_o) - \beta)}$$

where $F(t)$ was defined in (17). Note also this immediately implies $\tan \theta_o \geq 1$.

In this situation, due to the symmetry and convexity of ζ_1 and ζ_2 , no line thru P_o can intersect either curve more than once in any quadrant. Furthermore, as noted in the previous section, since $(2\beta + \eta) > 0$ it follows cycles with $c \geq 0$ occur only for $-1 \leq \rho < 1$, which equates to lines thru P_o whose slopes satisfy

$$- \tan \theta_o \leq m_\sigma \leq 0 ,$$

where θ_o was defined in Theorem 3 and Figure 6 .

The easiest way to arrive at the information needed to determine $\frac{d\rho}{d\sigma}$ is by describing the variation of m_1 and m_2 as σ increases from $-\theta_o$ to zero. In this case, observe that θ_o is determined by the line thru P_o to $(G(T_o), F(T_o))$, the end point of ζ_1 . Thus as σ increases to zero, m_1 increases monotonically (due to the convexity of

ζ_1) from -1 to a positive value less than

$$\frac{\sinh \beta\pi}{\cosh \beta\pi + 1} < 1 \; .$$

Since, as noted above, $\tan \theta_o \geq 1$, then as σ increases to zero, $m = \tan \sigma$ must increase monotonically from $-\tan \theta_o$ to zero. Note that since a line through P_o as located here cannot intersect ζ_1 tangentially (except perhaps at the endpoint), it follows

$$m_\sigma < m_1 \quad \text{for} \quad -\theta_o < \sigma \leq 0 \; .$$

Lastly, since $m_\sigma \leq -1$ at $-\theta_o$, then the line through P_o at $-\theta_o$ must intersect ζ_2 in the second quadrant. (Since here ζ_2 is asymtotic to a curve whose slope is greater than -1). Thus as σ increases, m_2 will initially monotonically decrease from a value in $(-1, 0)$ to

$$-\frac{\sinh \beta\pi}{\cosh \beta\pi + 1} \; ,$$

at which time the intersection point transitions to the fourth quadrant (and also ρ passes through zero, changing sign), then m_2 increases monotonically to a value at $\sigma = 0$ which is the negative of m_1 at $\sigma = 0$. It thus follows $m_\sigma < m_2 < 0$ in second quadrant, and $m_2 < m_\sigma < 0$ in the fourth quadrant for $\sigma \leq 0$.

If m_1, m_2 and m_σ are now plotted based on the above analysis, a picture like Figure 10 will emerge. Observe that m_1 and m_2 must intersect exactly once for $-\theta_o < \sigma < 0$. Now if we denote the intersection of m_1 and m_2 as $-\sigma_b$, and that of m_1 and m_σ as $-\sigma_t$, where necessarily $-\sigma_b < -\sigma_t$, we can easily construct Table 1 to give the algebraic signs necessary to interpret (33).

TABLE I

	$-\theta_o < \sigma < -\sigma_b$	$-\sigma_b < \sigma < -\sigma_t$	$-\sigma_t < \sigma < 0$
$m_1 - m_\sigma$	+	+	+
$m_2 - m_\sigma$	+	+	-
$m_2 - m_1$	+	-	-
ρ	+	+	-
$\frac{d\rho}{d\sigma}$	+	-	-

From this table it is clear that multiple cycles (for the same value of c) can exist in some cases. Furthermore this multiplicity can be of order exactly two, since $\frac{d\rho}{d\sigma}$ only changes sign once. (Again recall c is a monotonic function of ρ .) Lastly, the multiple cycles can occur only in a band of values of c whose lowest bound is strictly positive. In fact, due to the sign of $\frac{d\rho}{d\sigma}$ on $-\theta_o < \sigma < -\sigma_b$, the value of c at which the multiple cycles first occur is precisely the one found from the value of ρ for the line with slope $-\theta_o$. It should be noted that since the phase plane is two dimensional, when double cycles occur at least one will be unstable. This behavior, existence of a single cycle for $0 \leq c < c_b$, then two cycles for $c_b \leq c \leq c_{max}$, seems best described by the statement that a bifurcation occurs at c_b .

The situation when $(2\beta + \eta) \leq 0$ can be easily constructed in a manner to the above, and yields essentially the same result - a unique cycle will exist for $0 \leq |c| < c_b$ where $c_b < c_{max}$, there for $c_b \leq |c| \leq c_{max}$ exactly two cycles occur.

The other possible situations which arise become progressively more

complicated from a bookkeeping point of view. The reason for this lies in the geometry, with the convex nature of the curves involved. For, since ζ_1 is convex, the curve formed by ζ_1 plus the extension of the tangent line at the endpoint (at T_o) to the intersection with the verticle axis is also convex. Thus, where

$$- \frac{(1 + \beta^2)}{2(F(T_o) - \beta)} < 2\beta + \eta ,$$

P_o lies "outside" this extended curve, and hence for at least some angles σ, a line through P_o with slope m_σ will intersect ζ_1 twice. Thus m_1 cannot be described by a single-valued function of σ, and the analysis becomes more complex. However, as long as P_o is located below the line joining the endpoints of ζ_1 and ζ_2 in the third and fourth quadrants, the same basic behavior as was found in the simple cases above can be shown to exist.

However, when P_o lies on the line joining the endpoints of ζ_1 and ζ_2, i.e. when

$$0 < 2\beta + \eta = - \frac{(1 + \beta^2)}{F(T_o) - \beta} , \tag{34}$$

a qualitative change in the behavior occurs. For at this point, as Figure 11 shows, there are two solutions (points (1)-(3) and (2)-(4)) for which $\rho = -1$, and hence $c = 0$. Thus, in addition to the bifurcation at $c_b > 0$, there is now a second bifurcation at $c = 0$. As P_o continues to move toward $(0,-M)$, eventually a condition is reached where multiple cycles exist at all values of c.

Thus, in conclusions, we have shown that, in all cases, there is a c_b such that two cycles exist for $c_b \leq |c| \leq c_{max}$, and in certain cases

there is also a $c_b^{1} \leq c_b$ such that two cycles can exist for $0 \leq |c| \leq c_b^{1}$.

VI. SUMMARY

In this paper we have considered limit cycle solutions to

$$\dot{x}_1 = -2\beta x_1 - (\beta^2+1)x_2 + \alpha_1 \operatorname{sgn}(x_2-c)$$

$$\dot{x}_2 = x_1 + \alpha_2 \operatorname{sgn}(x_2-c) \quad .$$

We have derived the necessary and sufficient conditions for limit cycle solutions to exist, and have presented the conditions in terms of an equivalent, easily visualized, geometric condition. With the use of this geometric interpretation, we then showed that, with one exception, existence of a limit cycle for $c = 0$ is both necessary and sufficient for cycles to exist for some range of $c \neq 0$. Lastly, we have shown that multiple cycles, for some values of c , are possible is essentially all cases.

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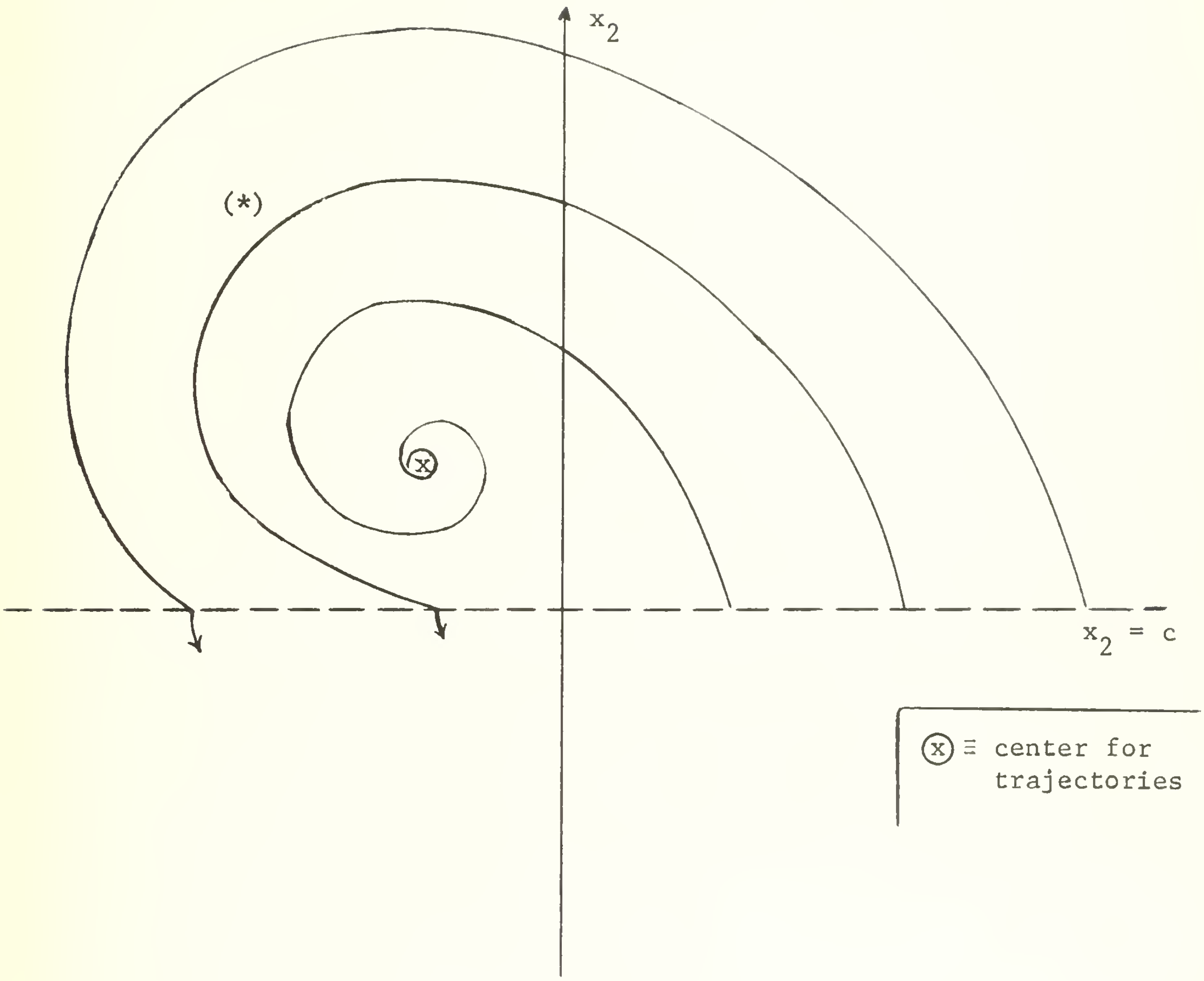
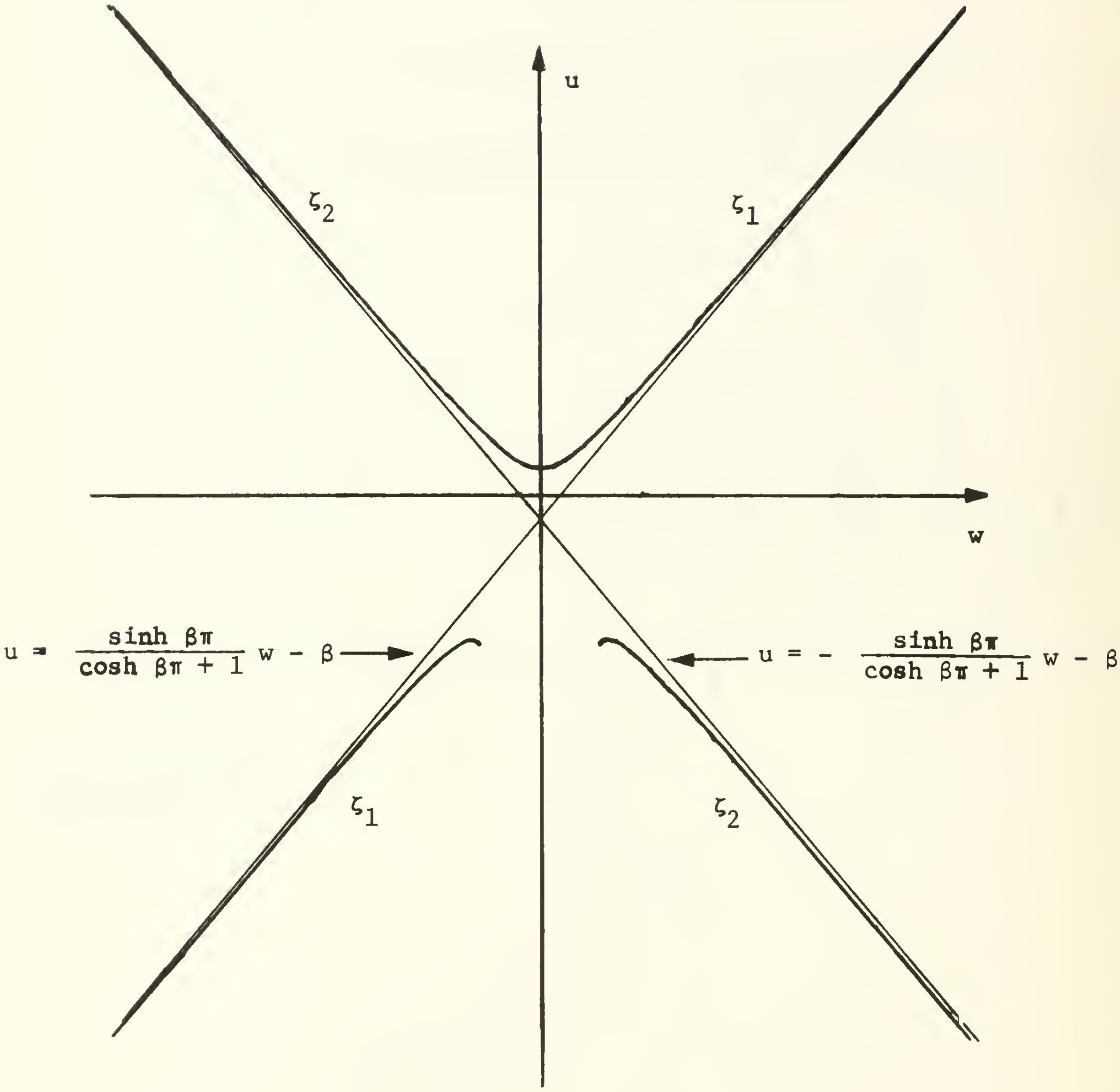


Figure 1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$u = \dfrac{\sinh \beta\pi}{\cosh \beta\pi + 1} w - \beta$
$u = - \dfrac{\sinh \beta\pi}{\cosh \beta\pi + 1} w - \beta$

Figure 2

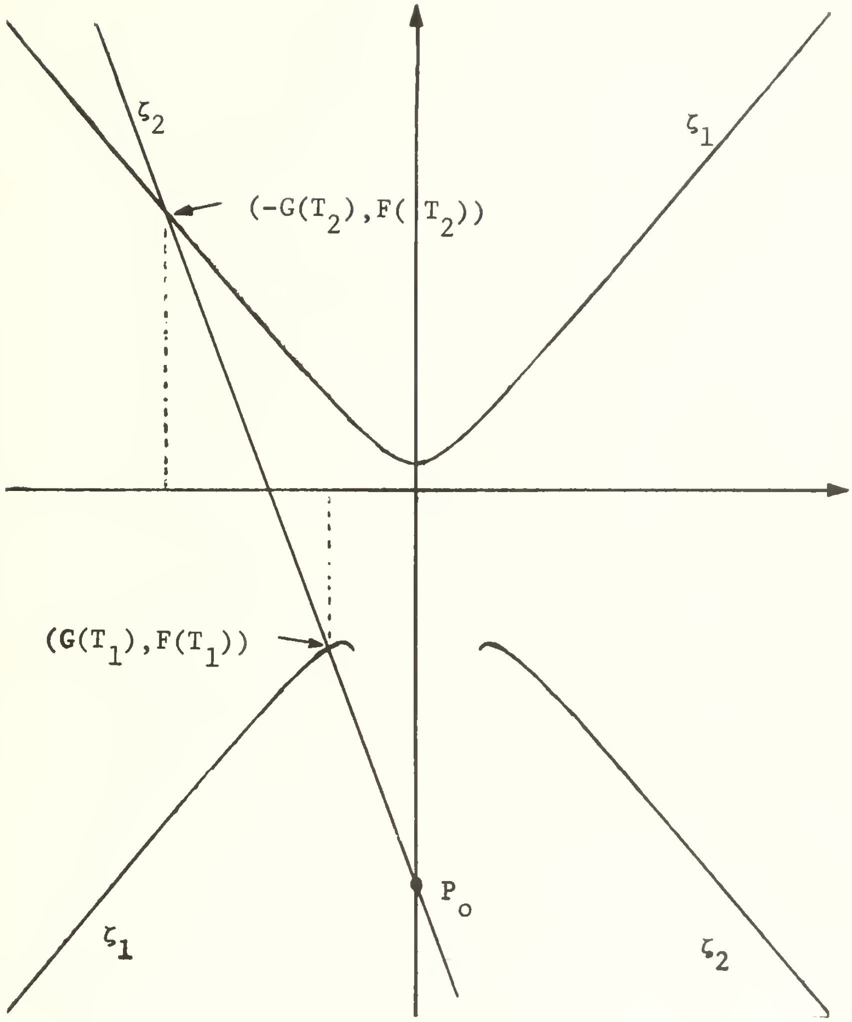
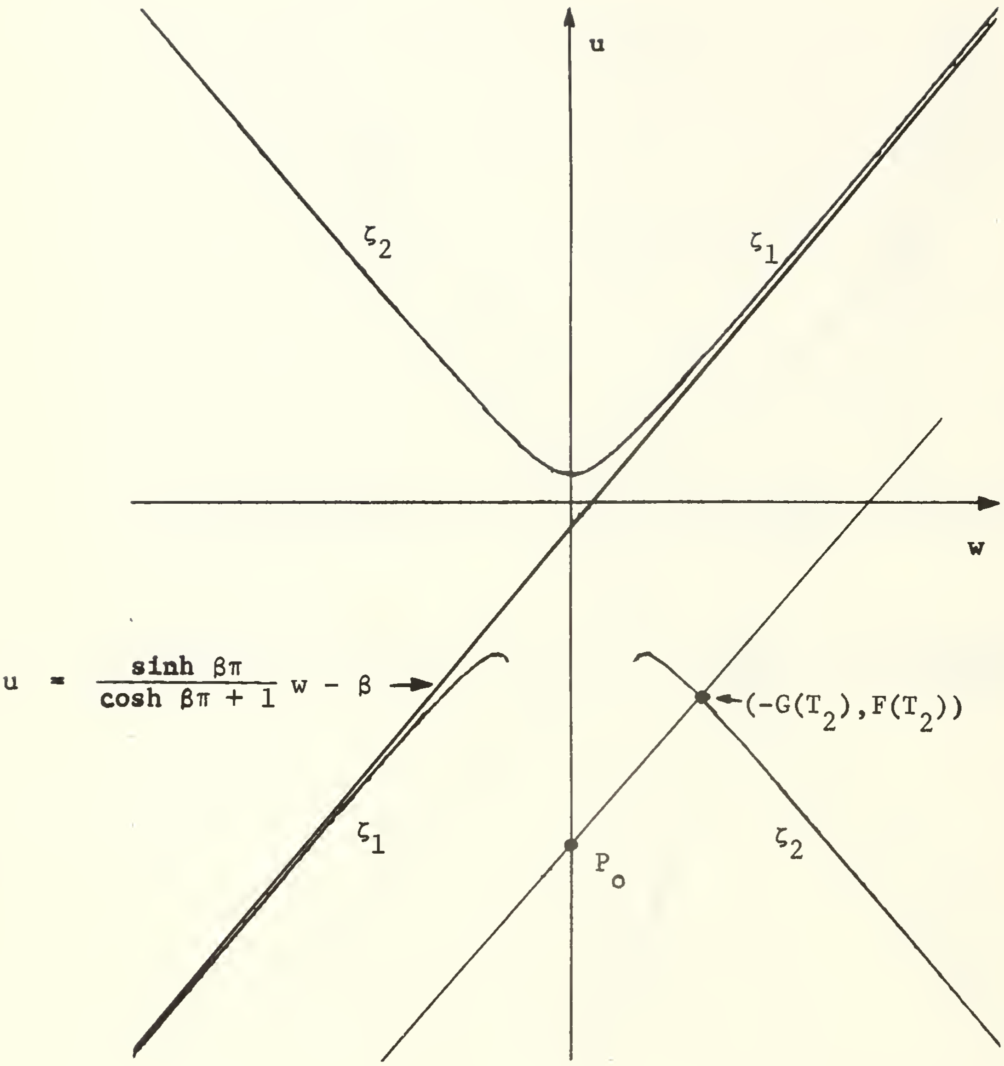
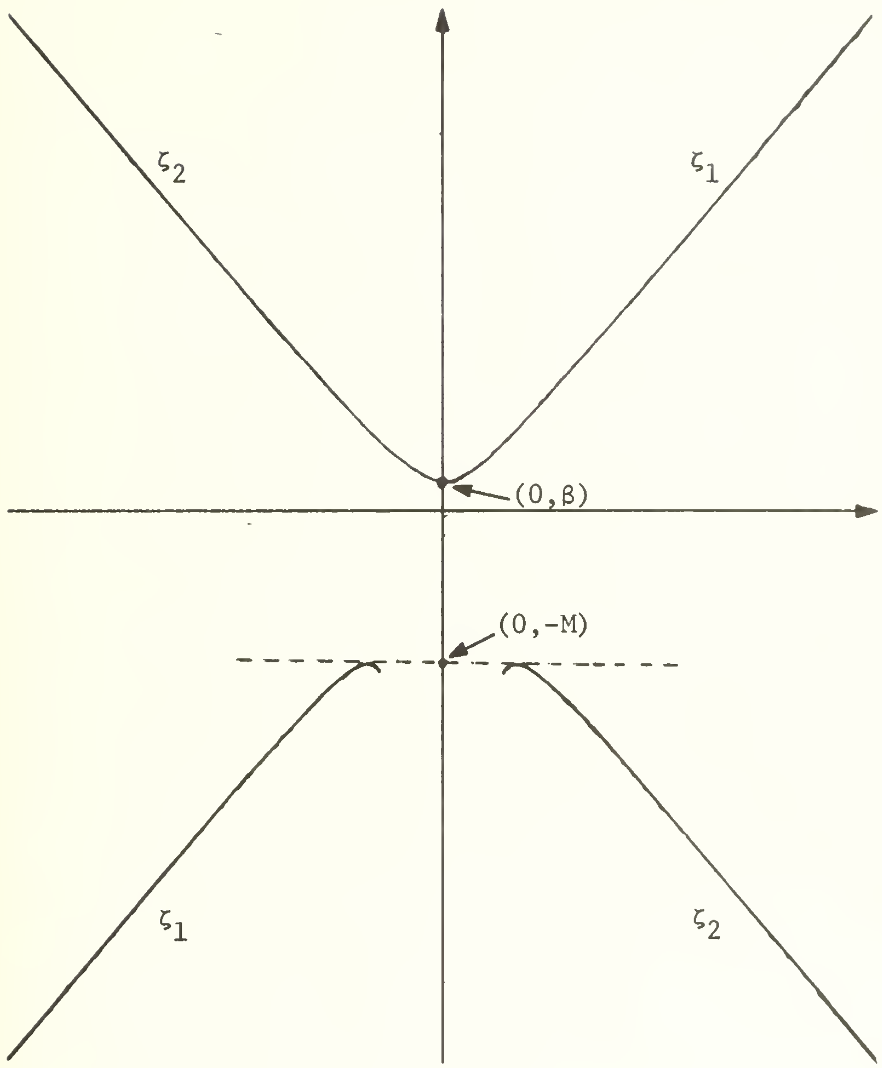


Figure 3



u
w
ζ_2
ζ_1
$u = \frac{\sinh \beta\pi}{\cosh \beta\pi + 1} w - \beta$
$(-G(T_2), F(T_2))$
ζ_1
P_o
ζ_2

Figure 4



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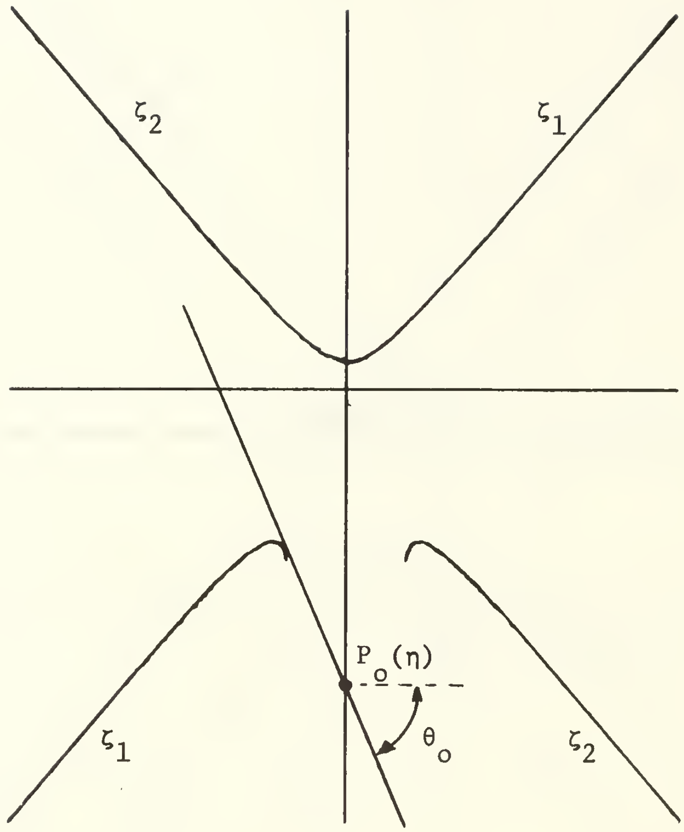

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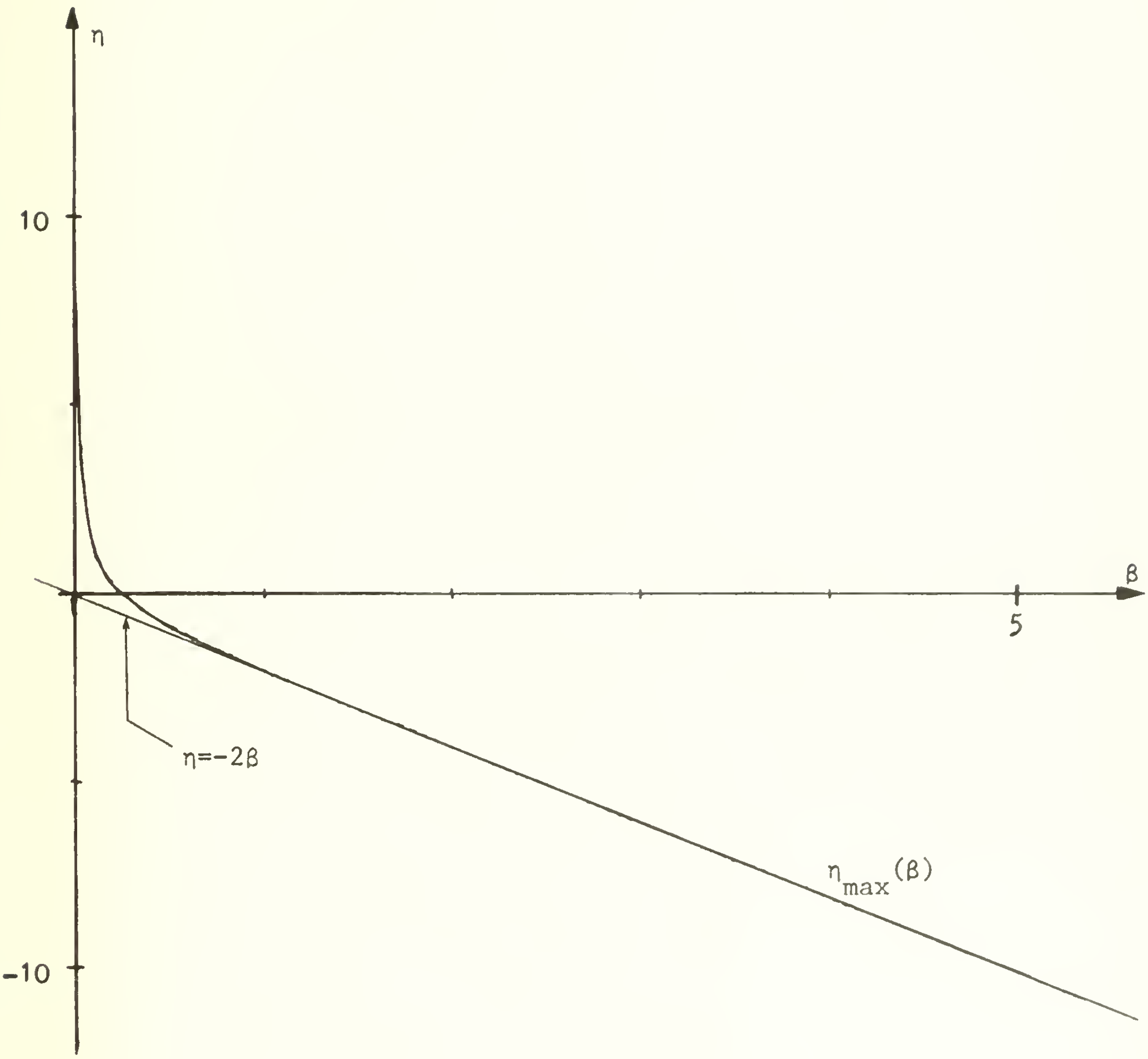

Figure 7

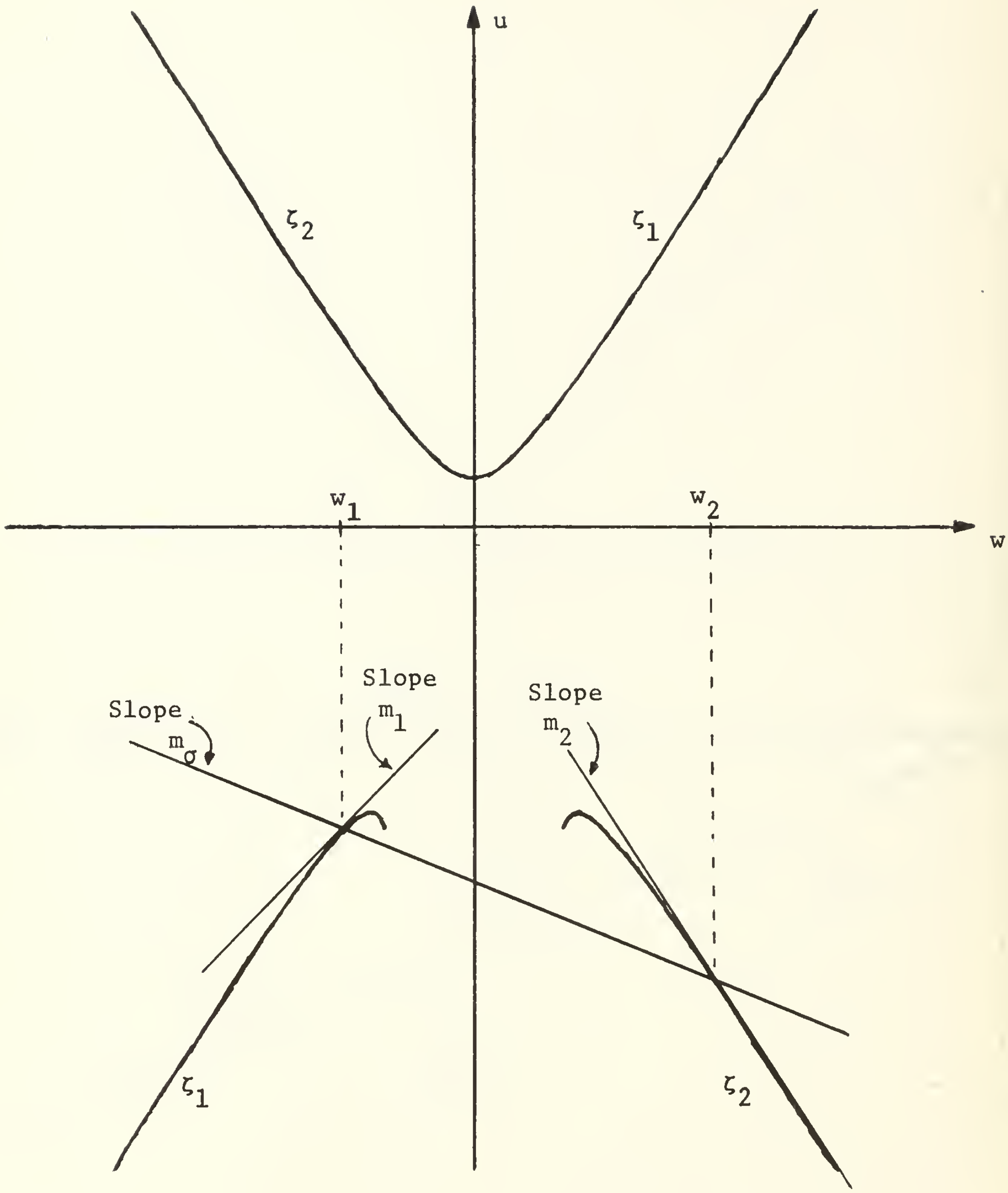
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Figure 8

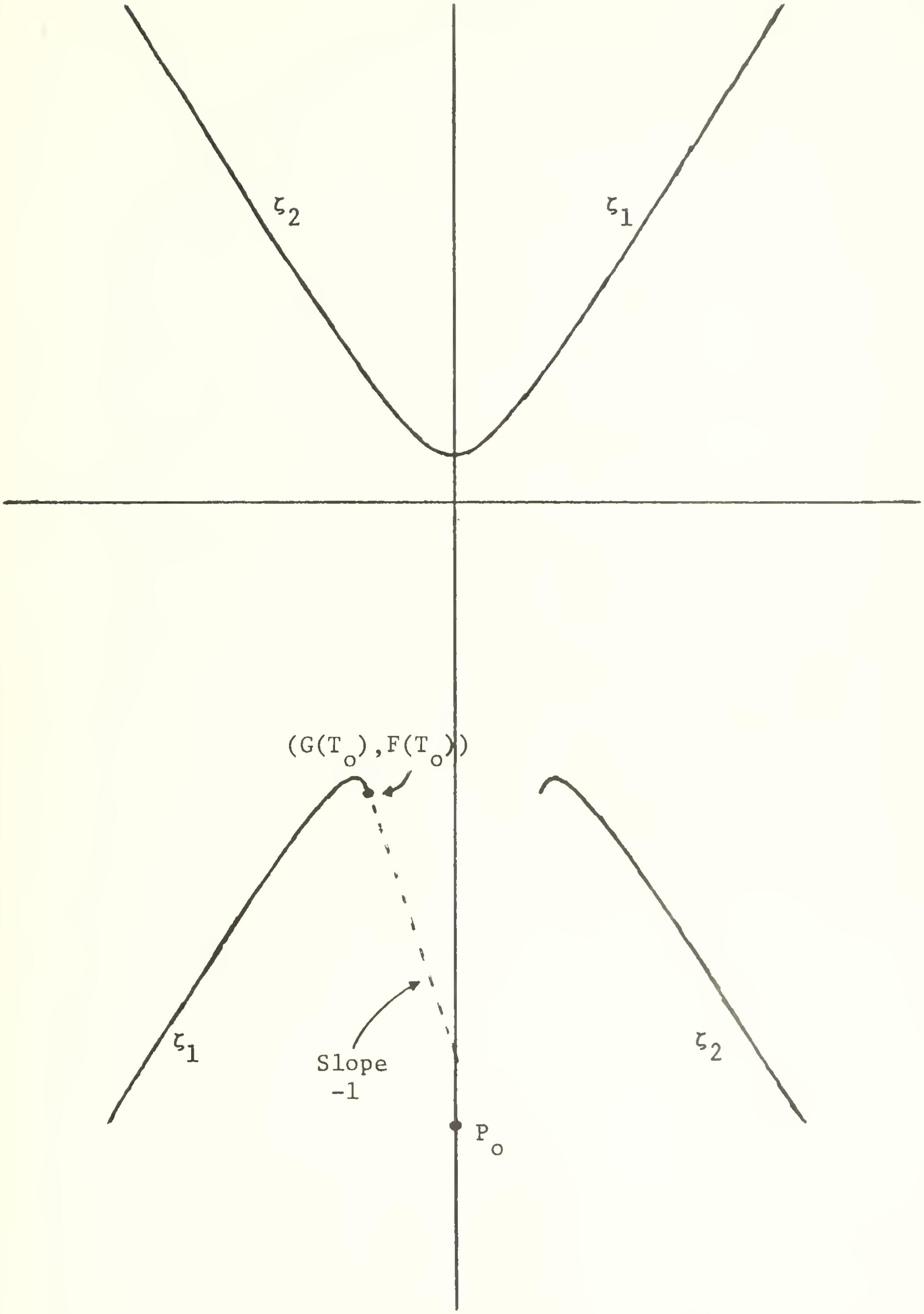

Figure 9

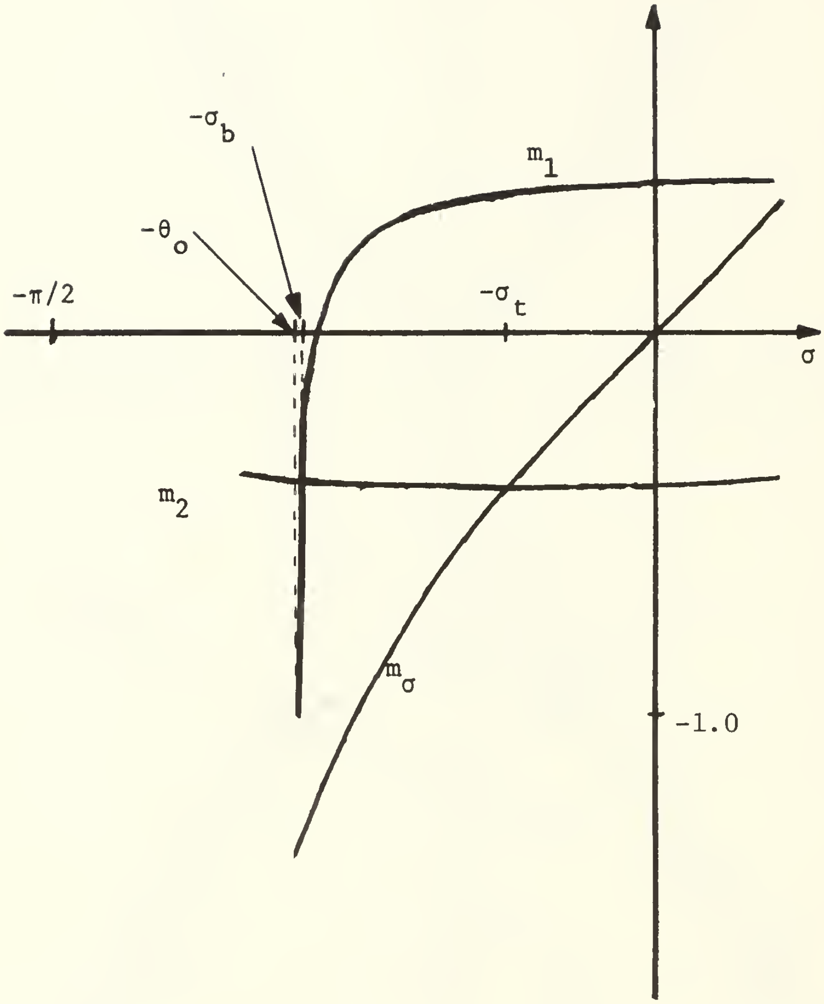

Figure 10

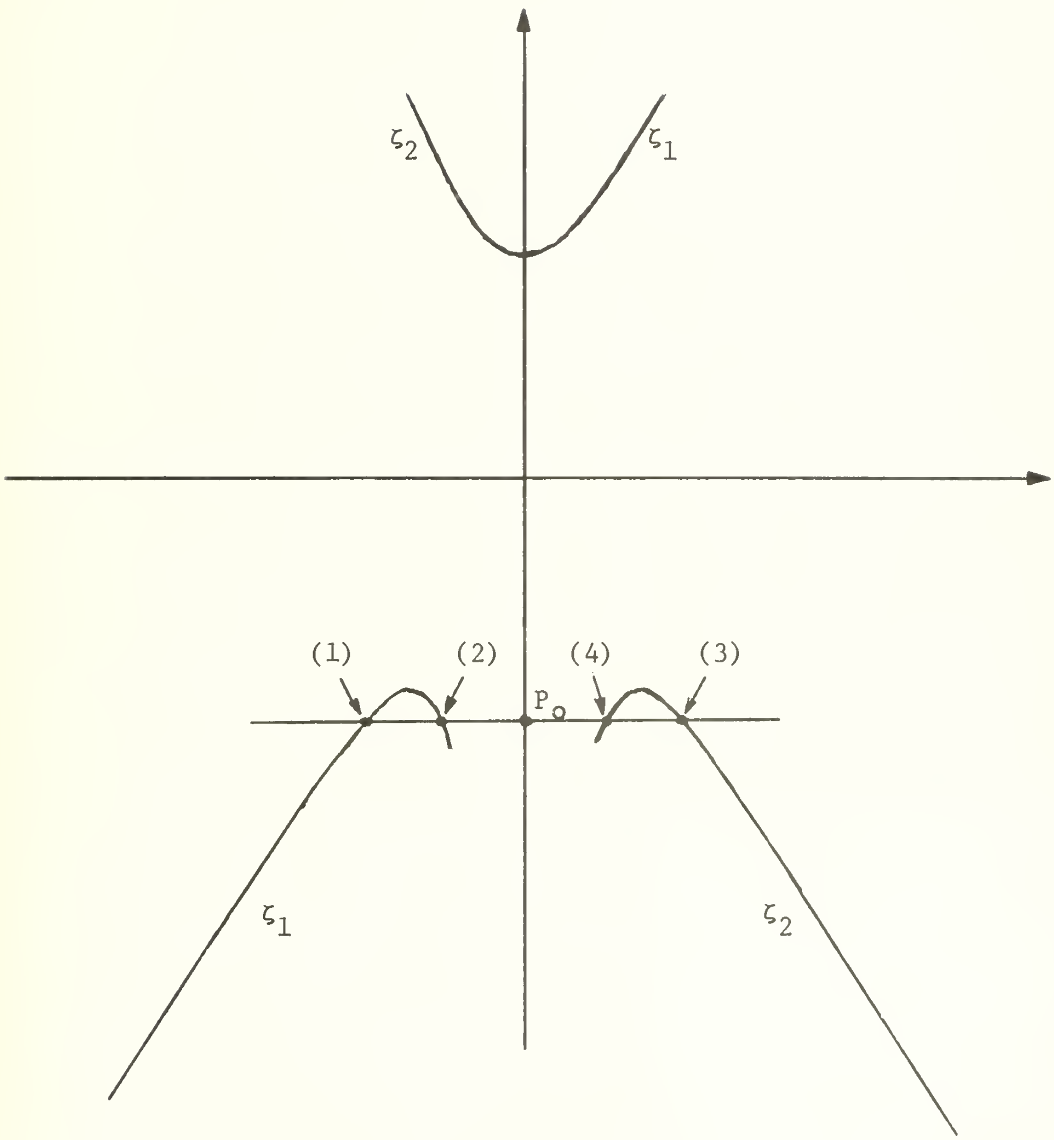

Figure 11

Appendix 1

We consider the system

$$\dot{\underline{v}} = \underline{\underline{A}}\,\underline{v} + \underline{u}_o \operatorname{sgn}(rv_1 + sv_2 - c) \tag{A-1}$$

where $\underline{u}_o$ is a constant vector, $r^2 + s^2 > 0$, and $\underline{\underline{A}}$ is a real, constant 2x2 matrix whose characteristic polynomial is $\lambda^2 + 2\gamma\lambda + (\gamma^2 + \omega^2)$, where $\gamma, \omega > 0$. First observe that the non-singular transformation

$$\underline{w} = \underline{\underline{B}}\,\underline{v}, \qquad \underline{\underline{B}} = \frac{1}{\omega}\begin{bmatrix} s & -r \\ r & s \end{bmatrix}$$

reduces (A-1) to

$$\dot{\underline{w}} = (\underline{\underline{B}}\,\underline{\underline{A}}\,\underline{\underline{B}}^{-1})\underline{w} + \underline{\underline{B}}\,\underline{u}_o \operatorname{sgn}(\omega w_2 - c) \quad , \tag{A-2}$$

where, because of similarily, $\underline{\underline{B}}\,\underline{\underline{A}}\,\underline{\underline{B}}^{-1}$ has the same characteristic polynomial as $\underline{\underline{A}}$, i.e.

$$\lambda^2 + 2\gamma\lambda + (\gamma^2 + \omega^2).$$

Since $\omega > 0$, both off-diagonal terms in $\underline{\underline{B}}\,\underline{\underline{A}}\,\underline{\underline{B}}^{-1}$ must be non-zero. and so $\underline{\underline{B}}\,\underline{\underline{A}}\,\underline{\underline{B}}^{-1}$ will have the form,

$$\underline{\underline{B}}\ \underline{\underline{A}}\ \underline{\underline{B}}^{-1} = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix},$$

where $d_{21}d_{12} \neq 0$. Thus $d_{21} \neq 0$, and hence the transformation

$$\underline{y} = \underline{\underline{C}}\ \underline{w}, \qquad \underline{\underline{C}} = \begin{bmatrix} d_{21} & d_{22} \\ 0 & 1 \end{bmatrix},$$

is non-singular. This transformation reduces (A-2) to

$$\dot{\underline{y}} = (\underline{\underline{C}}\ \underline{\underline{B}}\ \underline{\underline{A}}\ \underline{\underline{B}}^{-1}\ \underline{\underline{C}}^{-1})\ \underline{y} + \underline{\underline{C}}\ \underline{\underline{B}}\ \underline{u}_o\ \mathrm{sgn}\ (\omega y_2 - c) \tag{A-3}$$

where

$$\underline{\underline{C}}\ (\underline{\underline{B}}\ \underline{\underline{A}}\ B^{-1})\ \underline{\underline{C}}^{-1} = \begin{bmatrix} (d_{21} + d_{22}) & -(d_{11}d_{22} - d_{21}d_{12}) \\ 1 & 0 \end{bmatrix}$$

Again by similarity, this matrix has the characteristic polynomial

$$\lambda^2 + 2\gamma\lambda + (\gamma^2 + \omega^2).$$

By computing the characteristic polynomial directly, we see

$$-2\gamma = (d_{11} + d_{22})$$

$$\gamma^2 + \omega^2 = d_{11}d_{22} - d_{21}d_{12}$$

Thus,

$$\dot{y}_1 = -2\lambda y_1 - (\gamma^2 + \omega^2) y_2 + \tilde{u}_1 \operatorname{sgn}(\omega y_2 - c)$$

$$\dot{y}_2 = y_1 + \tilde{u}_2 \operatorname{sgn}(\omega y_2 - c) \tag{A-4}$$

where

$$\underline{\tilde{u}} = \underline{\underline{C}}\,\underline{\underline{B}}\,\underline{u}_o \quad .$$

But, now the time scaling $\omega t = \tau$ reduces (A-4) to

$$\dot{y}_1 = -2\left(\frac{\gamma}{\omega}\right) y_1 - \left(\left(\frac{\gamma}{\omega}\right)^2 + 1\right)(\omega y_2) + \frac{\tilde{u}_1}{\omega} \operatorname{sgn}((\omega y_2) - c)$$

$$(\omega \dot{y}_2) = y_1 + \tilde{u}_2 \operatorname{sgn}((\omega y_2) - c)$$

which, with the transformation

$$x_1 = y_1 , \quad x_2 = \omega y_2 , \quad \beta = \frac{\gamma}{\omega}$$

$$\alpha_1 = \frac{\tilde{u}_1}{\omega} , \quad \alpha_2 = \tilde{u}_2$$

becomes (2).

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